Semigroup Forum (2015) 90:113–125
DOI 10.1007/s00233-014-9573-4

RESEARCH ARTICLE

The relations of semiadjacency and semicompatibility in ∩-semigroups of transformations

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Received: 14 January 2013 / Accepted: 14 January 2014 / Published online: 12 December 2014

Abstract We consider semigroups of transformations (partial mappings defined on a set A) closed under the set-theoretic intersection of mappings treated as subsets of $A \times A$. On such semigroups we define two relations: the relation of semicompatibility which identifies two transformations at the intersection of their domains and the relation of semiadjacency when the image of one transformation is contained in the domain of the second. Abstract characterizations of such semigroups are presented.

Keywords Semigroup of transformations · Algebra of functions · Semiadjacency · Semicompatibility

1 Introduction

1. Let $\mathscr{F}(A)$ be the set of all transformations (i.e., the partial maps) of a non-empty set A. The domain of $f \in \mathscr{F}(A)$ is denoted by $\mathrm{pr}_1 f$, the image by $\mathrm{pr}_2 f$. The symbol $\Delta_{\mathrm{pr}_1 f}$ is reserved for the identity relation on $\mathrm{pr}_1 f$. The composition (superposition) of maps $f, g \in \mathscr{F}(A)$ is defined as $(g \circ f)(a) = g(f(a))$, where for every $a \in A$ the left and right hand side are defined, or undefined, simultaneously (cf. [1]). If the set $\Phi \subset \mathscr{F}(A)$ is closed with respect to such composition, then the algebra $(\Phi, \circ)$ is called a *semigroup of transformations* (cf. [1] or [10]). If Φ is also closed with respect

Communicated by Mikhail Volkov.

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to the set-theoretic intersection of transformations treated as subsets of $A \times A$, then the algebra $(\Phi, \circ, \cap)$ is called a $\cap$-*semigroup of transformations*.

On such a $\cap$-semigroup we can consider the so-called *semicompatibility relation* ξ_Φ defined as follows:

$$(f, g) \in \xi_\Phi \longleftrightarrow f \circ \Delta_{\mathrm{pr}_1 g} = g \circ \Delta_{\mathrm{pr}_1 f}. \tag{1}$$

The algebraic system $(\Phi, \circ, \cap, \xi_\Phi)$ is called a *transformative* $\cap$-*semigroup of transformations*. The investigation of such semigroups was initiated by Vagner [14] and continued by Saliĭ [7,8] and Schein [9]. A first abstract characterization of $\cap$-semigroups of transformations was found by Garvatskiĭ [4].

Some abstract characterizations of transformative $\cap$-semigroups of transformations can be deduced from results proved in [3] and [13] for Menger $\cap$-algebras of n-place functions.

On $(\Phi, \circ)$ we can also consider the *semiadjacency relation*

$$\delta_\Phi = \{(f, g) \mid \mathrm{pr}_2 f \subset \mathrm{pr}_1 g\}.$$

An abstract characterization of semigroups of transformations with this relation was established in [6]. Later, in [5], an abstract characterization of the algebraic system $(\Phi, \circ, \xi_\Phi, \delta_\Phi)$ was given, and in [2] $\cap$-semigroups of transformations with the semiadjacency relation were characterized. The semiadjacency relation on algebras of multiplace functions was investigated in [11].

In this paper we find an abstract characterization of $\cap$-semigroups of transformations equipped with the semicompatibility relation and the relation of semiadjacency.

We start with the following lemma.

Lemma 1 *The relation of semiadjacency defined on a transformation semigroup* $(\Phi, \circ)$ *satisfies the following two conditions:*

$$(f, g) \in \delta_\Phi \longleftrightarrow \mathrm{pr}_1 f \subset \mathrm{pr}_1 (g \circ f), \tag{2}$$

$$(f, g) \in \delta_\Phi \longrightarrow (f \circ h, g) \in \delta_\Phi. \tag{3}$$

We omit the proof of this lemma since it is a simple consequence of results proved in [2,5,6].

2. Each homomorphism P of an abstract semigroup $(G, \cdot)$ into the semigroup $(\mathscr{F}(A), \circ)$ of all transformations of a set A is called a *representation of* $(G, \cdot)$ *by transformations*. In the case when a representation is an isomorphism we say that it is *faithful*.

With each representation P of a semigroup $(G, \cdot)$ by transformations of A, we associate three binary relations on G:

$$\begin{aligned}
\zeta_P &= \{(g_1, g_2) \mid P(g_1) \subset P(g_2)\},\\
\xi_P &= \{(g_1, g_2) \mid P(g_1) \circ \Delta_{\mathrm{pr}_1 P(g_2)} = P(g_2) \circ \Delta_{\mathrm{pr}_1 P(g_1)}\},\\
\delta_P &= \{(g_1, g_2) \mid \mathrm{pr}_2 P(g_1) \subset \mathrm{pr}_1 P(g_2)\}.
\end{aligned}$$

Let $(P_i)_{i\in I}$ be a family of representations of a semigroup $(G,\cdot)$ by transformations of disjoint sets $(A_i)_{i\in I}$. By the *sum* of this family we mean the map $P\colon\ g\mapsto P(g)$, where $g\in G$, and $P(g)$ is the transformation on $A=\bigcup_{i\in I}A_i$ defined by $P(g)=\bigcup_{i\in I}P_i(g)$. It is easy to see that P is a representation of $(G,\cdot)$. It is denoted by $\sum_{i\in I}P_i$. If $P=\sum_{i\in I}P_i$, then obviously

$$\zeta_P=\bigcap_{i\in I}\zeta_{P_i},\quad \xi_P=\bigcap_{i\in I}\xi_{P_i},\quad \delta_P=\bigcap_{i\in I}\delta_{P_i}. \tag{4}$$

3. Following [1] and [10], we call a binary relation ρ on a semigroup $(G,\cdot)$:

- *stable* or *regular* if
  $(x,y)\in\rho\wedge(u,v)\in\rho\longrightarrow(xu,yv)\in\rho$ for all $x,y,u,v\in G$;
- *left regular* if $(u,v)\in\rho\longrightarrow(xu,xv)\in\rho$ for all $x,u,v\in G$;
- *right regular* if $(x,y)\in\rho\longrightarrow(xu,yu)\in\rho$ for all $x,y,u\in G$;
- *left ideal* if $(x,y)\in\rho\longrightarrow(ux,y)\in\rho$ for all $x,y,u\in G$;
- *right negative* $(x,yu)\in\rho\longrightarrow(x,y)\in\rho$ if for all $x,y,u\in G$.

A quasi-order ρ, i.e., a reflexive and transitive relation, is stable if and only if it is left and right regular (cf. [1,10]). Similarly, it is right negative if and only if $(xy,x)\in\rho$ for all $x,y\in G$.

Let $(G,\cdot)$ be an arbitrary semigroup, $(G^*,\cdot)$ the semigroup obtained from $(G,\cdot)$ by adjoining an identity $e\notin G$. By a *determining pair* of a semigroup $(G,\cdot)$ we mean an ordered pair (ε,W), where ε is a right regular equivalence relation on the semigroup $(G^*,\cdot)$, and W is either the empty set or an ε-class which is a right ideal of $(G,\cdot)$. Let $(H_a)_{a\in A}$ be the collection of all ε-classes (uniquely indexed by elements of A) such that $H_a\neq W$. As is well known (cf. [10]), with each determining pair (ε,W) one can associate the so-called *simplest representation* $P_{(\varepsilon,W)}$ of $(G,\cdot)$ by transformations defined in the following way:

$$(a_1,a_2)\in P_{(\varepsilon,W)}(g)\longleftrightarrow H_{a_1}g\subset H_{a_2}, \tag{5}$$

where $g\in G$, $a_1,a_2\in A$.

From results proved in [9] and [10] we can deduce the following properties of simplest representations.

Proposition 1 *Let (ε,W) be the determining pair of a semigroup $(G,\cdot)$. Then for all $g_1,g_2\in G$, $x\in G^*$ we have*

$$(g_1,g_2)\in\zeta_{P_{(\varepsilon,W)}}\longleftrightarrow(\forall x)(xg_1\notin W\longrightarrow xg_1\equiv xg_2(\varepsilon)), \tag{6}$$
$$(g_1,g_2)\in\xi_{P_{(\varepsilon,W)}}\longleftrightarrow(\forall x)(xg_1\notin W\wedge xg_2\notin W\longrightarrow xg_1\equiv xg_2(\varepsilon)), \tag{7}$$
$$(g_1,g_2)\in\delta_{P_{(\varepsilon,W)}}\longleftrightarrow(\forall x)(xg_1\notin W\longrightarrow xg_1g_2\notin W). \tag{8}$$

Proposition 2 *Suppose that $(G,\cdot,\curlywedge)$ is an algebraic system such that $(G,\cdot)$ is a semigroup, $(G,\curlywedge)$ is a semilattice and the identity*

$$x(y\curlywedge z)=xy\curlywedge xz, \tag{9}$$

holds. Then the equality

$$P_{(\varepsilon,W)}(g_1 \curlywedge g_2) = P_{(\varepsilon,W)}(g_1) \cap P_{(\varepsilon,W)}(g_2) \tag{10}$$

holds for arbitrary elements $g_1, g_2 \in G$ *and a determining pair* (ε, W) *of* $(G, \cdot)$ *if and only if*

$$g_1 \in W \longrightarrow g_1 \curlywedge g_2 \in W, \tag{11}$$

$$g_1 \curlywedge g_2 \notin W \longrightarrow g_1 \equiv g_2(\varepsilon), \tag{12}$$

$$g_1 \notin W \wedge g_1 \equiv g_2(\varepsilon) \longrightarrow g_1 \curlywedge g_2 \equiv g_1(\varepsilon). \tag{13}$$

An analogous result was proved in [12] (see also [3]) for Menger algebras of rank n. For $n = 1$ it gives the above proposition.

4. In this section we consider a *semilattice algebraic system* $(G, \cdot, \curlywedge, \xi, \delta)$, i.e., an algebraic system $(G, \cdot, \curlywedge, \xi, \delta)$ such that $(G, \cdot)$ is a semigroup, $(G, \curlywedge)$ is a semilattice, δ is a left ideal relation on $(G, \cdot)$, and ξ is a left regular binary relation on $(G, \cdot)$ containing the natural order ζ of the semilattice $(G, \curlywedge)$. (Recall that $(x, y) \in \zeta \longleftrightarrow x \curlywedge y = x$.) Assume that $(G, \cdot, \curlywedge, \xi, \delta)$ satisfies (9) as well as the conditions:

$$(x, y), (u, v) \in \zeta \ \wedge \ (y, v) \in \xi \longrightarrow (u, x) \in \xi, \tag{14}$$

$$(x, y) \in \xi \longrightarrow (x \curlywedge y)u = xu \curlywedge yu, \tag{15}$$

where $x, y, z, u, v \in G$. Moreover, we assume also that in the semigroup $(G^*, \cdot)$ with the adjoined identity e we have $(e, e) \in \zeta$, $(e, e) \in \delta$ and $(x, e) \in \delta$ for all $x \in G$.

Proposition 3 *If* $(G, \cdot, \curlywedge, \xi, \delta)$ *is a semilattice algebraic system, then the relation* ξ *is reflexive and symmetric and the relation* ζ *is stable on the semigroup* $(G, \cdot)$.

Proof The relation ξ is reflexive since $\zeta \subset \xi$ and ζ is the natural order on the semilattice $(G, \curlywedge)$. It also is symmetric because for every $(x, y) \in \xi$ we have $x\zeta x$, $y\zeta y$, and $x\xi y$, whence, by (14), we obtain $(y, x) \in \xi$.

To prove that ζ is stable on the semigroup $(G, \cdot)$ assume that $(x, y) \in \zeta$ for some $x, y \in G$. Then $x \curlywedge y = x$. Hence $z(x \curlywedge y) = zx$, which, by (9), gives $zx \curlywedge zy = zx$. Thus $(zx, zy) \in \zeta$. So, ζ is left regular. Since $\zeta \subset \xi$, from $(x, y) \in \zeta$, it follows $(x, y) \in \xi$, which, by (15), implies $(x \curlywedge y)z = xz \curlywedge yz$. Hence $xz = xz \curlywedge yz$, i.e., $(xz, yz) \in \zeta$. This means that ζ is right regular. Consequently, ζ is stable on the semigroup $(G, \cdot)$.

In the sequel, the formula $x\delta y \ \wedge \ xy\zeta z$ will be abbreviated as $x \boxdot y\zeta z$.

Definition 1 A subset $H \subset G$ is f_ξ*-closed* if the implication

$$(u, v) \in \xi \ \wedge \ (u \curlywedge v)x \boxdot y\zeta zt \ \wedge \ u, vx \in H \longrightarrow z \in H \tag{16}$$

holds true for all $x, y, t \in G^*$ and $z, u, v \in G$.

Clearly the set of all f_ξ-closed subsets of G forms a complete lattice under intersection. Given $X \subset G$, let $f_\xi(X)$ be the least f_ξ-closed subset of G containing X.

Proposition 4 *A non-empty subset H of a semilattice algebraic system $(G, \cdot, \curlywedge, \xi, \delta)$ is f_ξ-closed if and only if H satisfies the following conditions:*

$$xy \in H \longrightarrow x \in H, \tag{17}$$
$$(g_1, g_2) \in \delta \ \wedge \ g_1 \in H \longrightarrow g_1g_2 \in H, \tag{18}$$
$$g_1 \curlywedge g_2 = g_1 \in H \longrightarrow g_2 \in H, \tag{19}$$
$$(g_1, g_2) \in \xi \ \wedge \ g_1, g_2x \in H \longrightarrow (g_1 \curlywedge g_2)x \in H, \tag{20}$$

where x in (20) may be the empty symbol.

Proof Let H be an f_ξ-closed subset of G. Then

$$(u, v) \in \xi \ \wedge \ (u \curlywedge v)x\delta y \ \wedge \ (u \curlywedge v)xy\zeta zt \ \wedge \ u, vx \in H \longrightarrow z \in H \tag{21}$$

for all $x, y, t \in G^*$ and $z, u, v \in G$.

Using (21) we can prove conditions (17)–(20). Indeed, for $u = v = xy, x = y = e, t = y, z = x$ the implication (21) has the form

$$(xy, xy) \in \xi \ \wedge \ (xy \curlywedge xy)e\delta e \ \wedge \ (xy \curlywedge xy)e\zeta xy \ \wedge \ xy, xye \in H \longrightarrow x \in H.$$

Since relations ξ and ζ are reflexive and the operation $\curlywedge$ is idempotent, the last condition is equivalent to the implication (17).

For $u = v = g_1, x = e, y = g_1, t = e, z = g_1g_2$ the implication (21) gives the condition

$$(g_1, g_1) \in \xi \ \wedge \ (g_1 \curlywedge g_1)e\delta g_2 \ \wedge \ (g_1 \curlywedge g_1)eg_2\zeta g_1g_2e \ \wedge \ g_1, g_1e \in H \longrightarrow g_1g_2 \in H,$$

which is equivalent to (18).

Similarly for $u = v = g_1, x = y = t = e, z = g_2$ from (21) we obtain

$$(g_1, g_1) \in \xi \ \wedge \ (g_1 \curlywedge g_1)e\delta e \ \wedge \ (g_1 \curlywedge g_1)ee\zeta g_2e \ \wedge \ g_1, g_1e \in H \longrightarrow g_2 \in H,$$

i.e., $(g_1, g_2) \in \zeta \wedge g_1 \in H \longrightarrow g_2 \in H$. Thus, (21) implies (19).

Finally, (21) for $u = g_1, v = g_2, y = e, z = (g_1 \curlywedge g_2)x, t = e$, gives

$$(g_1, g_2) \in \xi \wedge (g_1 \curlywedge g_2)x\delta e \wedge (g_1 \curlywedge g_2)xe\zeta(g_1 \curlywedge g_2)xe \wedge g_1, g_2x \in H$$
$$\longrightarrow (g_1 \curlywedge g_2)x \in H,$$

which implies (20).

To prove the converse, assume that (17)–(20) and the premise of (21) are satisfied. Then from $(u, v) \in \xi \wedge u, vx \in H$, according to (20), we obtain $(u \curlywedge v)x \in H$. Since $(u \curlywedge v)x\delta y$, by (18), the last condition implies $(u \curlywedge v)xy \in H$. But $(u \curlywedge v)xy\zeta zt$, by (19), gives $zt \in H$, which by (17) gives $z \in H$. Thus, (17)–(20) imply (21).

For a non-empty subset H of G we define the set

$$F_\xi(H) = \{z \mid (\exists u, v, x, y, t)\ (u, v) \in \xi \wedge (u \curlywedge v)x \boxdot y\zeta zt \wedge u, vx \in H)\},$$

where $x, y, t \in G^*$ and $z, u, v \in G$.

Lemma 2 *For any subsets $H,\ H_1,\ H_2$ of G we have*

(a) $H \subset F_\xi(H)$,
(b) $F_\xi(H_1) \subset F_\xi(H_2)$ *for* $H_1 \subset H_2$.
(c) $F_\xi(H) = H$ *for every* f_ξ*-closed subset* H *of* G.

Proof Indeed, if $z \in H$, then

$$(z, z) \in \xi \wedge (z \curlywedge z)e \boxdot e\zeta ze \wedge z, ze \in H,$$

which means that $z \in F_\xi(H)$. Hence, $H \subset F_\xi(H)$.

The second claim is obvious.

To prove the last claim, assume that H is an f_ξ-closed subset of G. Then for every $z \in F_\xi(H)$ and some $x, y, t \in G^*$, $u, v \in G$ we have

$$(u, v) \in \xi \wedge (u \curlywedge v)x \boxdot y\zeta zt \wedge u, vx \in H.$$

Since H is f_ξ-closed, the above implies $z \in H$. Thus $F_\xi(H) \subset H$, which together with (a) proves $F_\xi(H) = H$.

Given a non-empty subset $H \subset G$, we put $\overset{0}{F}_\xi(H) = H$ and $\overset{n}{F}_\xi(H) = F_\xi\Big(\overset{n-1}{F}_\xi(H)\Big)$ for every positive integer n. Then, by Lemma 2, we have

$$H = \overset{0}{F}_\xi(H) \subset \overset{1}{F}_\xi(H) \subset \overset{2}{F}_\xi(H) \subset \overset{3}{F}_\xi(H) \subset \ldots$$

Proposition 5 *Let $(G, \cdot, \curlywedge, \xi, \delta)$ be a semilattice algebraic system, H a non-empty subset of G. Then*

$$f_\xi(H) = \bigcup_{n=0}^{\infty} \overset{n}{F}_\xi(H). \tag{22}$$

Proof Let $\overline{H}_\xi = \bigcup\limits_{n=0}^{\infty} \overset{n}{F}_\xi(H)$ and

$$(u, v) \in \xi \wedge (u \curlywedge v)x\delta y \wedge (u \curlywedge v)xy\zeta zt \wedge u, vx \in \overline{H}_\xi\ ,$$

for some $x, y, t \in G^*$ and $z, u, v \in G$. Since $u, vx \in \overline{H}_\xi$, there are natural numbers n_1, n_2 such that $u \in \overset{n_1}{F}_\xi(H)$ and $vx \in \overset{n_2}{F}_\xi(H)$. Hence $\overset{n_i}{F}_\xi(H) \subset \overset{n}{F}_\xi(H)$, $i = 1, 2$, for $n = \max(n_1, n_2)$. Therefore

$$(u, v) \in \xi \wedge (u \curlywedge v)x \boxdot y\zeta zt \wedge u, vx \in \overset{n}{F}_{\xi}(H),$$

so, $z \in \overset{n+1}{F}_{\xi}(H) \subset \overline{H}_{\xi}$. This proves that $\overline{H}_{\xi}$ is a f_{ξ}-closed subset of G.

By the definition $H \subset f_{\xi}(H)$. Hence, by Lemma 2, $F_{\xi}(H) \subset F_{\xi}(f_{\xi}(H)) = f_{\xi}(H)$. Similarly, $\overset{2}{F}_{\xi}(H) \subset f_{\xi}(H)$, etc. Consequently, $\overset{n}{F}_{\xi}(H) \subset f_{\xi}(H)$ for any n, which implies $\bigcup_{n=0}^{\infty} \overset{n}{F}_{\xi}(H) \subset f_{\xi}(H)$, i.e., $\overline{H}_{\xi} \subset f_{\xi}(H)$. On the other hand, $H \subset \bigcup_{n=0}^{\infty} \overset{n}{F}_{\xi}(H) = \overline{H}_{\xi}$. Therefore $f_{\xi}(H) \subset f_{\xi}(\overline{H}_{\xi}) = \overline{H}_{\xi}$. Thus $\overline{H}_{\xi} = f_{\xi}(H)$, which proves (22).

Using a straightforward induction, we can easily prove the following proposition.

Proposition 6 *For each subset H of a semilattice algebraic system $(G, \cdot, \curlywedge, \xi, \delta)$, every natural number $n > 1$ and each $z \in G$, we have $z \in \overset{n}{F}_{\xi}(H)$ if and only if for some $x_i, y_i, t_i \in G^*$ and $u_i, v_i \in G$ the following system of conditions holds true:*

$$\left(\begin{array}{c} (u_1, v_1) \in \xi \wedge (u_1 \curlywedge v_1)x_1 \boxdot y_1\zeta zt_1, \\ \bigwedge\limits_{i=1}^{2^{n-1}-1} \left(\begin{array}{l} (u_{2i}, v_{2i}) \in \xi \wedge (u_{2i} \curlywedge v_{2i})x_{2i} \boxdot y_{2i}\zeta u_i t_{2i}, \\ (u_{2i+1}, v_{2i+1}) \in \xi \wedge (u_{2i+1} \curlywedge v_{2i+1})x_{2i+1} \boxdot y_{2i+1}\zeta v_i x_i t_{2i+1} \end{array} \right), \\ \bigwedge\limits_{i=2^{n-1}}^{2^n-1} (u_i, v_i x_i \in H). \end{array} \right)$$

In the sequel the system of the above conditions will be denoted by $\mathfrak{X}_n(z, H)$.

5. Let $(\Phi, \circ, \cap, \xi_{\Phi}, \delta_{\Phi})$ be a transformative ∩-semigroup of transformations with the relation of semicompatibility ξ_{Φ} and the relation of semiadjacency δ_{Φ}.

Proposition 7 $\bigcap_{\varphi_i \in H_{\Phi}} \mathrm{pr}_1\varphi_i \subset \mathrm{pr}_1\varphi$ *for every $H_{\Phi} \subset \Phi$ and $\varphi \in f_{\xi_{\Phi}}(H_{\Phi})$.*

Proof First we show that the following implication

$$\varphi \in \overset{n}{F}_{\xi_{\Phi}}(H_{\Phi}) \longrightarrow \bigcap_{\varphi_i \in H_{\Phi}} \mathrm{pr}_1\varphi_i \subset \mathrm{pr}_1\varphi \tag{23}$$

is valid for every integer n. We prove it by induction.

Let $\mathfrak{A} = \bigcap_{\varphi_i \in H_{\Phi}} \mathrm{pr}_1\varphi_i$. If $n = 0$ and $\varphi \in \overset{0}{F}_{\xi_{\Phi}}(H_{\Phi})$, then clearly $\varphi \in H_{\Phi}$. Thus $\mathfrak{A} \subset \mathrm{pr}_1\varphi$, which verifies (23) for $n = 0$.

Assume now that (23) is valid for some $n > 0$. To prove that it is valid for $n + 1$, consider an arbitrary transformation $\varphi \in \overset{n+1}{F}_{\xi_{\Phi}}(H_{\Phi})$. Then, for some transformations $x, y, t, u, v \in \Phi$, where x, y, t may be the empty symbols, we have $(u, v) \in \xi_{\Phi}$, $(x \circ$

$(u \cap v), y) \in \delta_\Phi$, $y \circ x \circ (u \cap v) \subset t \circ \varphi$ and $u, x \circ v \in \overset{n}{F}_{\xi_\Phi}(H_\Phi)$. The last condition, according to the assumption on n, implies $\mathfrak{A} \subset \mathrm{pr}_1 u$. Similarly, $\mathfrak{A} \subset \mathrm{pr}_1(x \circ v) \subset \mathrm{pr}_1 v$. Consequently $\Delta_{\mathrm{pr}_1 u} \circ \Delta_{\mathfrak{A}} = \Delta_{\mathfrak{A}}$ and $\Delta_{\mathrm{pr}_1 v} \circ \Delta_{\mathfrak{A}} = \Delta_{\mathfrak{A}}$.

From $(x \circ (u \cap v), y) \in \Delta_\Phi$ it follows $\mathrm{pr}_2(x \circ (u \cap v)) \subset \mathrm{pr}_1 y$, which, by (2), gives $\mathrm{pr}_1(x \circ (u \cap v)) \subset \mathrm{pr}_1(y \circ x \circ (u \cap v)) \subset \mathrm{pr}_1(t \circ \varphi)$. Then, $(u, v) \in \xi_\Phi$ means that $u \circ \Delta_{\mathrm{pr}_1 v} = v \circ \Delta_{\mathrm{pr}_1 u}$. So, $u \circ \Delta_{\mathrm{pr}_1 v} \circ \Delta_{\mathfrak{A}} = v \circ \Delta_{\mathrm{pr}_1 u} \circ \Delta_{\mathfrak{A}}$, hence $u \circ \Delta_{\mathfrak{A}} = v \circ \Delta_{\mathfrak{A}} = u \circ \Delta_{\mathfrak{A}} \cap v \circ \Delta_{\mathfrak{A}} = (u \cap v) \circ \Delta_{\mathfrak{A}}$. Since $\mathfrak{A} \subset \mathrm{pr}_1(x \circ v)$, we have

$$
\begin{aligned}
\mathfrak{A} &\subset \mathrm{pr}_1(x \circ v \circ \Delta_{\mathfrak{A}}) = \mathrm{pr}_1(x \circ (u \cap v) \circ \Delta_{\mathfrak{A}}) \subset \mathrm{pr}_1(y \circ x \circ (u \cap v) \circ \Delta_{\mathfrak{A}}) \\
&\subset \mathrm{pr}_1(t \circ \varphi \circ \Delta_{\mathfrak{A}}) \subset \mathrm{pr}_1(\varphi \circ \Delta_{\mathfrak{A}}) = \mathrm{pr}_1(\varphi \circ \Delta_{\mathrm{pr}_1 \varphi} \circ \Delta_{\mathfrak{A}}) \\
&= \mathrm{pr}_1(\varphi \circ \Delta_{\mathfrak{A}} \circ \Delta_{\mathrm{pr}_1 \varphi}) \subset \mathrm{pr}_1 \varphi.
\end{aligned}
$$

Thus, $\mathfrak{A} \subset \mathrm{pr}_1 \varphi$. This shows that (23) is valid for $n + 1$. Consequently, (23) is valid for all integers n.

To complete the proof of this proposition observe now that, according to (22), for every $\varphi \in f_{\xi_\Phi}(H_\Phi)$ there exists n such that $\varphi \in \overset{n}{F}_{\xi_\Phi}(H_\Phi)$, which, by (23), gives $\bigcap\limits_{\varphi_i \in H_\Phi} \mathrm{pr}_1 \varphi_i \subset \mathrm{pr}_1 \varphi$.

Theorem 1 *An algebraic system $(G, \cdot, \curlywedge, \xi, \delta)$, where $(G, \cdot)$ is a semigroup, $(G, \curlywedge)$ is a semilattice, ξ, δ are binary relations on G, is isomorphic to some transformative $\cap$-semigroup of transformations $(\Phi, \circ, \cap, \xi_\Phi, \delta_\Phi)$ if and only if ξ is a left regular relation containing the semilattice order ζ, δ is a left ideal relation on $(G, \cdot)$ and conditions (9), (14), (15), as well as the conditions:*

$$x \curlywedge y \in f_\xi(\{x\}) \longrightarrow x \zeta y, \tag{24}$$

$$x \curlywedge y \in f_\xi(\{x, y\}) \longrightarrow x \xi y, \tag{25}$$

$$xy \in f_\xi(\{x\}) \longrightarrow x \delta y \tag{26}$$

are satisfied by all elements of G.

Proof NECESSITY. Let $(\Phi, \circ, \cap, \xi_\Phi, \delta_\Phi)$ be a transformative $\cap$-semigroup of transformations of some set. We show that it satisfies all the conditions of our theorem.

The necessity of (9) is a consequence of results proved in [1] and [4]. Since the order ζ_Φ of the semilattice $(\Phi, \cap)$ coincides with the inclusion, ζ_Φ is contained in ξ_Φ. From (3) (Lemma 1) it follows that δ_Φ is a left ideal relation.

Let $(f, g) \in \xi_\Phi$, i.e., $f \circ \Delta_{\mathrm{pr}_1 g} = g \circ \Delta_{\mathrm{pr}_1 f}$. Then $f \circ \Delta_{\mathrm{pr}_1 g} \circ h = g \circ \Delta_{\mathrm{pr}_1 f} \circ h$. Since $\Delta_{\mathrm{pr}_1 g} \circ h = h \circ \Delta_{\mathrm{pr}_1 g \circ h}$ and $\Delta_{\mathrm{pr}_1 f} \circ h = h \circ \Delta_{\mathrm{pr}_1 f \circ h}$, we have $f \circ h \circ \Delta_{\mathrm{pr}_1 g \circ h} = g \circ h \circ \Delta_{\mathrm{pr}_1 f \circ h}$, which proves $(f \circ h, g \circ h) \in \xi_\Phi$. Thus, ξ_Φ is left regular.

If $f \subset g$, $h \subset p$ and $(g, p) \in \xi_\Phi$ for some $f, g, h, p \in \Phi$, then $f = g \circ \Delta_{\mathrm{pr}_1 f}$, $h = p \circ \Delta_{\mathrm{pr}_1 h}$ and $g \circ \Delta_{\mathrm{pr}_1 p} = p \circ \Delta_{\mathrm{pr}_1 g}$. The last equality implies $g \circ \Delta_{\mathrm{pr}_1 p} \circ \Delta_{\mathrm{pr}_1 f} \circ \Delta_{\mathrm{pr}_1 h} = p \circ \Delta_{\mathrm{pr}_1 g} \circ \Delta_{\mathrm{pr}_1 f} \circ \Delta_{\mathrm{pr}_1 h}$. Thus, $p \circ \Delta_{\mathrm{pr}_1 h} \circ \Delta_{\mathrm{pr}_1 g} \circ \Delta_{\mathrm{pr}_1 f} = g \circ \Delta_{\mathrm{pr}_1 f} \circ \Delta_{\mathrm{pr}_1 p} \circ \Delta_{\mathrm{pr}_1 h}$. Consequently, $h \circ \Delta_{\mathrm{pr}_1 g} \circ \Delta_{\mathrm{pr}_1 f} = f \circ \Delta_{\mathrm{pr}_1 p} \circ \Delta_{\mathrm{pr}_1 h}$, which in view of $\Delta_{\mathrm{pr}_1 g} \circ \Delta_{\mathrm{pr}_1 f} = \Delta_{\mathrm{pr}_1 f}$ and $\Delta_{\mathrm{pr}_1 p} \circ \Delta_{\mathrm{pr}_1 h} = \Delta_{\mathrm{pr}_1 h}$ gives $h \circ \Delta_{\mathrm{pr}_1 f} = f \circ \Delta_{\mathrm{pr}_1 h}$. Therefore, $(h, f) \in \xi_\Phi$. So, (14) is satisfied.

To prove (15) let $(f, g) \in \xi_\Phi$, i.e., $f \circ \Delta_{\mathrm{pr}_1 g} = g \circ \Delta_{\mathrm{pr}_1 f}$. Since

$$f \cap g = (f \cap g) \circ \Delta_{\mathrm{pr}_1 g} = f \circ \Delta_{\mathrm{pr}_1 g} \cap g = g \circ \Delta_{\mathrm{pr}_1 f} \cap g = g \circ \Delta_{\mathrm{pr}_1 f} \\ = f \circ \Delta_{\mathrm{pr}_1 g},$$

we have

$$h \circ (f \cap g) = h \circ f \circ \Delta_{\mathrm{pr}_1 g} \cap h \circ g \circ \Delta_{\mathrm{pr}_1 f} = (h \circ f \cap h \circ g) \circ \Delta_{\mathrm{pr}_1 g} \circ \Delta_{\mathrm{pr}_1 f} = \\ h \circ f \circ \Delta_{\mathrm{pr}_1 f} \cap h \circ g \circ \Delta_{\mathrm{pr}_1 g} = h \circ f \cap h \circ g.$$

Thus $h \circ (f \cap g) = h \circ f \cap h \circ g$, which proves (15).

Now let $\varphi \cap \psi \in f_{\xi_\Phi}(\{\varphi\})$ for some $\varphi, \psi \in \Phi$. Then $\mathrm{pr}_1\varphi \subset \mathrm{pr}_1(\varphi \cap \psi)$, by Proposition 7. Hence $\mathrm{pr}_1(\varphi \cap \psi) = \mathrm{pr}_1\varphi$ since $\mathrm{pr}_1(\varphi \cap \psi) \subset \mathrm{pr}_1\varphi$. Thus $\varphi = \varphi \circ \Delta_{\mathrm{pr}_1\varphi} = \varphi \circ \Delta_{\mathrm{pr}_1(\varphi\cap\psi)} = \varphi \cap \psi \subset \psi$. This proves (24), because the inclusion $\subset$ coincides with the order ζ_Φ of the semilattice $(\Phi, \cap)$.

If $\varphi \cap \psi \in f_{\xi_\Phi}(\{\varphi, \psi\})$, then, by Proposition 7, $\mathrm{pr}_1\varphi \cap \mathrm{pr}_1\psi \subset \mathrm{pr}_1(\varphi \cap \psi)$, which together with the obvious inclusion $\mathrm{pr}_1(\varphi \cap \psi) \subset \mathrm{pr}_1\varphi \cap \mathrm{pr}_1\psi$ gives $\mathrm{pr}_1(\varphi \cap \psi) = \mathrm{pr}_1\varphi \cap \mathrm{pr}_1\psi$. So,

$$\varphi \circ \Delta_{\mathrm{pr}_1\psi} = \varphi \circ \Delta_{\mathrm{pr}_1\varphi} \circ \Delta_{\mathrm{pr}_1\psi} = \varphi \circ \Delta_{\mathrm{pr}_1\varphi\cap\,\mathrm{pr}_1\psi} = \varphi \circ \Delta_{\mathrm{pr}_1(\varphi\cap\psi)} = \varphi \cap \psi = \\ \psi \circ \Delta_{\mathrm{pr}_1(\varphi\cap\psi)} = \psi \circ \Delta_{\mathrm{pr}_1\psi\cap\,\mathrm{pr}_1\varphi} = \psi \circ \Delta_{\mathrm{pr}_1\psi} \circ \Delta_{\mathrm{pr}_1\varphi} = \psi \circ \Delta_{\mathrm{pr}_1\varphi}.$$

Thus $\varphi \circ \Delta_{\mathrm{pr}_1\psi} = \psi \circ \Delta_{\mathrm{pr}_1\varphi}$, i.e., $(\varphi, \psi) \in \xi_\Phi$. This proves (25).

To prove the last condition let $\psi \circ \varphi \in f_{\xi_\Phi}(\{\varphi\})$. Then $\mathrm{pr}_1\varphi \subset \mathrm{pr}_1(\psi \circ \varphi)$, which by (2), gives $(\varphi, \psi) \in \delta_\Phi$. This means that (26) also is satisfied.

Sufficiency. Let $(G, \cdot, \curlywedge, \xi, \delta)$ be an algebraic system satisfying all the conditions of the theorem. Then, by Proposition 3, ξ is a reflexive and symmetric relation, and ζ is stable in the semigroup $(G, \cdot)$. Moreover, the implication

$$(g_1, g_2) \in \zeta \wedge g_1 \in f_\xi(\{x, y\}) \longrightarrow g_2 \in f_\xi(\{x, y\}) \tag{27}$$

holds true for all $g_1, g_2, x, y \in G$. In fact, the premise of (27) can be rewritten in the form:

$$(g_1, g_1) \in \xi \wedge (g_1 \curlywedge g_1)e \boxdot e\zeta g_2 e \wedge g_1, g_1 e \in f_\xi(\{x, y\}).$$

So, if it is satisfied, then, according to the definition of $F_\xi(H)$ and Lemma 2, $g_2 \in F_\xi(f_\xi(\{x, y\})) = f_\xi(\{x, y\})$, which proves (27).

Now we show that for all $x, y \in G$ the subset $G \setminus f_\xi(\{x, y\})$ is a right ideal of the semigroup $(G, \cdot)$. Indeed, if $gu \in f_\xi(\{x, y\})$, then, by (22), for some natural n we have $gu \in \overset{n}{F}_\xi(\{x, y\})$. Hence

$$(gu, gu) \in \xi \wedge (gu \curlywedge gu)e \boxdot e\zeta gu \wedge gu, gue \in \overset{n}{F}_\xi(\{x, y\}),$$

so, $g \in \overset{n+1}{F}_{\xi}(\{x, y\}) \subset f_{\xi}(\{x, y\})$. Thus, $g \in f_{\xi}(\{x, y\})$. In this way we have shown the implication $gu \in f_{\xi}(\{x, y\}) \longrightarrow g \in f_{\xi}(\{x, y\})$, which by the contraposition is equivalent to the implication $g \notin f_{\xi}(\{x, y\}) \longrightarrow gu \notin f_{\xi}(\{x, y\})$. The last implication means that $G \setminus f_{\xi}(\{x, y\})$ is a right ideal.

If $(u, v) \in \xi$ for $u, v \in f_{\xi}(\{x, y\})$, then, obviously,

$$(u, v) \in \xi \wedge (u \curlywedge v)e\delta e \wedge (u \curlywedge v)ee\zeta(u \curlywedge v)e \wedge u, ve \in f_{\xi}(\{x, y\}).$$

Thus $u \curlywedge v \in F_{\xi}(f_{\xi}(\{x, y\})) = f_{\xi}(\{x, y\})$, since the set $f_{\xi}(\{x, y\})$ is f_{ξ}-closed. So, $f_{\xi}(\{x, y\})$ satisfies the implication

$$(u, v) \in \xi \wedge u, v \in f_{\xi}(\{x, y\}) \longrightarrow u \curlywedge v \in f_{\xi}(\{x, y\}). \tag{28}$$

We show now that the relation

$$\varepsilon_{(g_1,g_2)} = \{(x, y) \mid x \curlywedge y \in f_{\xi}(\{g_1, g_2\}) \vee x, y \notin f_{\xi}(\{g_1, g_2\})\}$$

defined on the semigroup $(G, \cdot)$ is a right regular equivalence and $G \setminus f_{\xi}(\{g_1, g_2\})$ is an equivalence class.

The reflexivity and symmetry of $\varepsilon_{(g_1,g_2)}$ are obvious. To prove the transitivity let $(x, y), (y, z) \in \varepsilon_{(g_1,g_2)}$. If $x, y, z \notin f_{\xi}(\{g_1, g_2\})$, then clearly $(x, z) \in \varepsilon_{(g_1,g_2)}$. In the case $x \curlywedge y \in f_{\xi}(\{g_1, g_2\})$ from $x \curlywedge y\zeta y$, by (27), we conclude $y \in f_{\xi}(\{g_1, g_2\})$. Therefore $x, z \in f_{\xi}(\{g_1, g_2\})$. Consequently, $x \curlywedge y,\ y \curlywedge z \in f_{\xi}(\{g_1, g_2\})$. But $(x \curlywedge y)\zeta y$, $(y \curlywedge z)\zeta y$ and $y\xi y$, hence the last, by (14), implies $(x \curlywedge y)\xi(y \curlywedge z)$. From this, applying (28), we deduce $x \curlywedge y \curlywedge z \in f_{\xi}(\{g_1, g_2\})$. On the other hand $(x \curlywedge y \curlywedge z)\zeta(x \curlywedge z)$ for all $x, y, z \in G$. So, $x \curlywedge y \curlywedge z \in f_{\xi}(\{g_1, g_2\})$, according to (27), implies $x \curlywedge z \in f_{\xi}(\{g_1, g_2\})$. Hence $(x, z) \in \varepsilon_{(g_1,g_2)}$. This proves the transitivity of $\varepsilon_{(g_1,g_2)}$. Summarizing $\varepsilon_{(g_1,g_2)}$ is an equivalence relation.

If $x, y \in G \setminus f_{\xi}(\{g_1, g_2\})$, then we have $(x, y) \in \varepsilon_{(g_1,g_2)}$. This means that the subset $G \setminus f_{\xi}(\{g_1, g_2\})$ is contained in some $\varepsilon_{(g_1,g_2)}$-class. Now let $x \in G \setminus f_{\xi}(\{g_1, g_2\})$ and $(x, y) \in \varepsilon_{(g_1,g_2)}$. The case $x \curlywedge y \in f_{\xi}(\{g_1, g_2\})$ is impossible, because in this case $x \in f_{\xi}(\{g_1, g_2\})$. So, $y \notin f_{\xi}(\{g_1, g_2\})$, i.e., $y \in G \setminus f_{\xi}(\{g_1, g_2\})$. Hence $G \setminus f_{\xi}(\{g_1, g_2\})$ coincides with some $\varepsilon_{(g_1,g_2)}$-class.

To prove that the relation $\varepsilon_{(g_1,g_2)}$ is right regular, we take a pair $(x, y) \in \varepsilon_{(g_1,g_2)}$. If $x, y \in G \setminus f_{\xi}(\{g_1, g_2\})$, then $xz, yz \in G \setminus f_{\xi}(\{g_1, g_2\})$ since $G \setminus f_{\xi}(\{g_1, g_2\})$ is a right ideal. Thus $(xz, yz) \in f_{\xi}(\{g_1, g_2\})$. Now if $x \curlywedge y, xz \in f_{\xi}(\{g_1, g_2\})$, then

$$(x \curlywedge y, x) \in \xi \wedge (x \curlywedge y)z\delta e \wedge (x \curlywedge y)ze\zeta(x \curlywedge y)ze \wedge (x \curlywedge y), xz \in f_{\xi}(\{g_1, g_2\}),$$

whence, by (16), we obtain $(x \curlywedge y)z \in f_{\xi}(\{g_1, g_2\})$. But $(x \curlywedge y)z\zeta yz$, whence we get $yz \in f_{\xi}(\{g_1, g_2\})$. Similarly, from $x \curlywedge y \in f_{\xi}(\{g_1, g_2\})$ and $yz \in f_{\xi}(\{g_1, g_2\})$ we get $xz \in f_{\xi}(\{g_1, g_2\})$. So, if $x \curlywedge y \in f_{\xi}(\{g_1, g_2\})$, then xz, yz belong or do not belong to $f_{\xi}(\{g_1, g_2\})$ simultaneously. If $xz, yz \notin f_{\xi}(\{g_1, g_2\})$, then obviously, $(xz, yz) \in \varepsilon_{(g_1,g_2)}$. If $xz, yz \in f_{\xi}(\{g_1, g_2\})$, then, as was shown above, from $x \curlywedge y \in f_{\xi}(\{g_1, g_2\})$ it follows $(x \curlywedge y)z \in f_{\xi}(\{g_1, g_2\})$. Since $(x \curlywedge y)z\zeta xz$ and $(x \curlywedge y)z\zeta yz$, then obviously

$(x \curlywedge y)z\zeta(xz \curlywedge yz)$. Hence $xz \curlywedge yz \in f_\xi(\{g_1, g_2\})$, i.e., $(xz, yz) \in \varepsilon_{(g_1,g_2)}$. So, in any case $(x, y) \in \varepsilon_{(g_1,g_2)}$ implies $(xz, yz) \in \varepsilon_{(g_1,g_2)}$. This proves that $\varepsilon_{(g_1,g_2)}$ is right regular.

From what was just shown, it follows that the pair $(\varepsilon^*_{(g_1,g_2)}, W_{(g_1,g_2)})$, where

$$\varepsilon^*_{(g_1,g_2)} = \varepsilon_{(g_1,g_2)} \cup \{(e, e)\}, \qquad W_{(g_1,g_2)} = G \setminus f_\xi(\{g_1, g_2\}),$$

is a determining pair of the semigroup $(G, \cdot)$.

Let $\big(P_{(\varepsilon^*_{(g_1,g_2)}, W_{(g_1,g_2)})}\big)_{(g_1,g_2)\in G\times G}$ be the family of simplest representations of the semigroup $(G, \cdot)$. Their sum

$$P = \sum_{(g_1,g_2)\in G\times G} P_{(\varepsilon^*_{(g_1,g_2)}, W_{(g_1,g_2)})} \tag{29}$$

is a representation of $(G, \cdot)$ by transformations. It is easy to see that the above determining pairs satisfy (11)–(13). Therefore, by Proposition 2, we have

$$P_{(\varepsilon^*_{(g_1,g_2)}, W_{(g_1,g_2)})}(x \curlywedge y) = P_{(\varepsilon^*_{(g_1,g_2)}, W_{(g_1,g_2)})}(x) \cap P_{(\varepsilon^*_{(g_1,g_2)}, W_{(g_1,g_2)})}(y)$$

for all $g_1, g_2 \in G$. Hence $P(x \curlywedge y) = P(x) \cap P(y)$ for $x, y \in G$. Thus, P is a homomorphism of the algebra $(G, \cdot, \curlywedge)$ onto the ∩-semigroup $(\Phi, \circ, \cap)$, where $\Phi = P(G)$.

Now we prove that $\xi = \xi_P$ and $\delta = \delta_P$. In fact, according to (4) and (7) we have

$$\begin{aligned}&(x, y) \in \xi_P \quad \longleftrightarrow \bigcap_{(g_1,g_2)\in G\times G} \xi_{(\varepsilon^*_{(g_1,g_2)}, W_{(g_1,g_2)})} \quad \longleftrightarrow\\ &(\forall g_1)(\forall g_2)(\forall u \in G^*)\big(ux, uy \in f_\xi(\{g_1, g_2\}) \longrightarrow ux \curlywedge uy \in f_\xi(\{g_1, g_2\})\big).\end{aligned}$$

The last implication for $u = e$ and $g_1 = x$, $g_2 = y$ has the form

$$x, y \in f_\xi(\{x, y\}) \longrightarrow x \curlywedge y \in f_\xi(\{x, y\}).$$

Thus $x \curlywedge y \in f_\xi(\{x, y\})$. Hence, by (25), we obtain $x\xi y$. This proves $\xi_P \subset \xi$.

To prove the converse inclusion, let $(x, y) \in \xi$. If $ux, uy \in f_\xi(\{g_1, g_2\})$ for some $u \in G^*$ and $g_1, g_2 \in G$, then from $(x, y) \in \xi$, by the left regularity of ξ, we obtain $(ux, uy) \in \xi$, which by (28) implies $ux \curlywedge uy \in f_\xi(\{g_1, g_2\})$. Therefore $(ux, uy) \in \xi_{(\varepsilon^*_{(g_1,g_2)}, W_{(g_1,g_2)})}$. Thus $(x, y) \in \bigcap_{(g_1,g_2)\in G\times G} \xi_{(\varepsilon^*_{(g_1,g_2)}, W_{(g_1,g_2)})} = \xi_P$. So, $\xi \subset \xi_P$ and $\xi = \xi_P$.

Now if $(x, y) \in \delta$ and $ux \in f_\xi(\{g_1, g_2\})$ for some $g_1, g_2 \in G$ and $u \in G^*$, then also $(ux, y) \in \delta$ because δ is a left ideal of $(G, \cdot)$. Since $f_\xi(\{g_1, g_2\})$ is f_ξ-closed, the condidion $(ux, y) \in \delta$ together with $ux \in f_\xi(\{g_1, g_2\})$, according to (18), implies that $uxy \in f_\xi(\{g_1, g_2\})$. Thus $(x, y) \in \delta_{(\varepsilon^*_{(g_1,g_2)}, W_{(g_1,g_2)})}$. Hence we conclude that $(x, y) \in \bigcap_{(g_1,g_2)\in G\times G} \delta_{(\varepsilon^*_{(g_1,g_2)}, W_{(g_1,g_2)})} = \delta_P$, and this proves $\delta \subset \delta_P$.

Conversely, let $(x, y) \in \delta_P$. Then, in view of (4) and (8), we have

$$(\forall g_1)\big(\forall g_2)(\forall u \in G^*)(ux \in f_\xi(\{g_1, g_2\}) \longrightarrow uxy \in f_\xi(\{g_1, g_2\})\big),$$

which for $u = e$ and $g_1 = g_2 = x$ has the form

$$x \in f_\xi(\{x\}) \longrightarrow xy \in f_\xi(\{x\}).$$

Thus $xy \in f_\xi(\{x\})$. This, by (26), implies $(x, y) \in \delta$. So, $\delta_P \subset \delta$, and hence $\delta_P = \delta$.

In this way we have shown that P is a homomorphism of $(G, \cdot, \curlywedge, \xi, \delta)$ onto the $\cap$-semigroup $(\Phi, \circ, \cap, \xi_\Phi, \delta_\Phi)$, where $\Phi = P(G)$.

It is also an isomorphism. To prove this fact observe first that $\zeta_P \subset \zeta$. Indeed, according to (4) and (6), we have:

$$(x, y) \in \zeta_P \;\longleftrightarrow \bigcap_{(g_1, g_2) \in G \times G} \zeta_{(\varepsilon^*_{(g_1, g_2)}, W_{(g_1, g_2)})} \;\longleftrightarrow$$
$$(\forall g_1)(\forall g_2)(\forall u \in G^*)\big(ux \in f_\xi(g_1, g_2) \longrightarrow ux \curlywedge uy \in f_\xi(\{g_1, g_2\})\big).$$

Putting $u = e$ and $g_1 = g_2 = x$ in the last implication, we obtain

$$x \in f_\xi(\{x\}) \longrightarrow x \curlywedge y \in f_\xi(\{x\}).$$

So, $x \curlywedge y \in f_\xi(\{x\})$. This, by (24), gives $x\zeta y$, i.e., $(x, y) \in \zeta$. Hence $\zeta_P \subset \zeta$.

Now let $P(g_1) = P(g_2)$. Then $P(g_1) \subset P(g_2)$ and $P(g_2) \subset P(g_1)$. Hence $(g_1, g_2) \in \zeta_P$ and $(g_2, g_1) \in \zeta_P$. This implies $(g_1, g_2), (g_2, g_1) \in \zeta$. Thus $g_1 = g_2$ because ζ is a semilattice order. So, P is an isomorphism between $(G, \cdot, \curlywedge, \xi, \delta)$ and $(\Phi, \circ, \cap, \xi_\Phi, \delta_\Phi)$.

Now, using (22) and the formula $\mathfrak{X}_n(z, H)$ from Proposition 6, we can write conditions (24)–(26) in the form of systems of elementary axioms $(A_n)_{n\in\mathbb{N}}$, $(B_n)_{n\in\mathbb{N}}$ and $(C_n)_{n\in\mathbb{N}}$, respectively, where

$$\begin{aligned} A_n\colon\ & \mathfrak{X}_n(x \curlywedge y, \{x\}) \longrightarrow x \curlywedge y = x,\\ B_n\colon\ & \mathfrak{X}_n(x \curlywedge y, \{x, y\}) \longrightarrow (x, y) \in \xi,\\ C_n\colon\ & \mathfrak{X}_n(xy, \{x\}) \longrightarrow (x, y) \in \delta. \end{aligned}$$

Thus, we have proved the following theorem:

Theorem 2 *An algebraic system* $(G, \cdot, \curlywedge, \xi, \delta)$*, where* $(G, \cdot)$ *is a semigroup,* $(G, \curlywedge)$ *is a semilattice,* ξ, δ *are binary relations on* G*, is isomorphic to some transformative* $\cap$*-semigroup of transformations* $(\Phi, \circ, \cap, \xi_\Phi, \delta_\Phi)$ *if and only if* ξ *is a left regular relation containing the semilattice order* ζ, δ *is a left ideal relation on* $(G, \cdot)$*, and the conditions* (9)*,* (14)*,* (15)*, as well as the axiom systems* $(A_n)_{n\in\mathbb{N}}$*,* $(B_n)_{n\in\mathbb{N}}$ *and* $(C_n)_{n\in\mathbb{N}}$ *are satisfied by all elements of* G*.*

The relation of semicompatibility and the relation of semiadjacency in a semigroup of transformations can be characterized by essentially infinite systems of elementary axioms (for details see [5,6,9]). Probably the axiom systems $(A_n)_{n\in\mathbb{N}}$, $(B_n)_{n\in\mathbb{N}}$, $(C_n)_{n\in\mathbb{N}}$ are also essentially infinite, i.e., they are not equivalent to any finite subsystems, but this problem requires further investigation.

Acknowledgments The authors are highly grateful to the anonymous referees and M. V. Volkov for their valuable comments and suggestions for improving the paper.

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